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RECENT PROGRESS IN THE DEVELOPMENT AND UNDERSTANDING OF SUPG METHODS WITH SPECIAL REFERENCE TO THE COMPRESSIBLE EULER AND NAVIER–STOKES EQUATIONS*†

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SUMMARY

SUPG methods were originally developed for the scalar advection–diffusion equation and the incompressible Navier–Stokes equations. In the last few years successful extensions have been made to symmetric advective–diffusive systems and, in particular, the compressible Euler and Navier–Stokes equations. New procedures have been introduced to improve resolution of discontinuities and thin layers. In this paper a brief overview is presented of recent progress in the development and understanding of SUPG methods.

KEY WORDS Advection–diffusion equation Advective–diffusive systems Artificial–diffusion Compressible flows Discontinuous Galerkin method Entropy Error analysis Euler equations Finite elements Galerkin method Hyperbolic systems Incompressible flow Navier–Stokes equations Petrov–Galerkin method Space–time formulation Upwind methods Weighted residual methods

INTRODUCTION

This paper gives an account of the current status of 'SUPG', a class of finite element methods which has proven effective on a variety of flow problems. The acronym I coined stands for 'streamline–upwind/Petrov–Galerkin'. The prominence paid the word 'streamline' is in retrospect unfortunate because for all applications except the scalar advection–diffusion equation, streamlines do not play the essential role. The use of the word 'upwind' was also a poor choice because of its pejorative connotations in some circles and because SUPG *really* is different from classical upwind methods which sacrifice accuracy in favour of stability by adding large doses of artificial diffusivity. SUPG *combines* higher-order accuracy with good stability properties. This has been exhibited in numerous calculations and proven mathematically. The term 'Petrov–Galerkin' is used these days to indicate that the method is *any* weighted residual method other than the classical Galerkin method. The use of Petrov's name seems to emanate from a reference in Mikhlin[1]. Based on this single contribution, it seems inappropriate to give Petrov the credit (and blame!) for *every* method not specifically Galerkin's. SUPG is a *particular* non-classical type of weighted residual method. At least Galerkin's name does seem appropriate: SUPG *starts* with classical Galerkin formulations

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and attempts to modify them to achieve improved behaviour. I accept the blame for inventing a terrible name for a good method. Some other researchers have proposed different names: Claes Johnson prefers 'streamline diffusion' and Shohei Nakazawa likes 'anisotropic balancing dissipation'. I think the words 'diffusion' and 'dissipation' are at least as bad as 'upwind'. I have already indicated why 'streamline' is inappropriate and I really do not think 'anisotropic balancing...' is any better. I suggest all names so far are more or less equally terrible (with apologies to my distinguished friends Claes Johnson and Shohei Nakazawa). For better or worse 'SUPG' seems to have stuck. The acronym is acceptable, but not what it stands for. I conjecture that many more people know of 'SUPG' than 'streamline'.... Perhaps this is as it should be. When the Stanford Research Institute separated from Stanford University it adopted the name SRI International. SRI is the *name*; it is not an *acronym* for Stanford Research Institute (at least that is what everybody says).

My initial paper on this subject, co-authored by Alec Brooks, was published in 1979 [2]. We were attempting to put Raithby's 'skew–upwind differencing' ideas into a finite element format. We recognized that the procedure lacked a rigorous weighted residual interpretation. The method was described for the scalar advection–diffusion equation and incompressible Navier–Stokes equations[2–4]. Based on a presentation of A. J. Baker[5] in May of 1979, I was aware of the connection of the approach with earlier works of Dendy[6], Wahlbin[7] and Raymond and Garter[8]. Kelly *et al.* described a similar approach in 1980[9]. Nakazawa continued along this path in his thesis research on coupled thermal-fluid polymer problems and in subsequent papers written in collaboration with his colleagues at Swansea[10–13]. (See also Argyris *et al.*[14] for applications to coupled thermal-fluid problems.) In all the early papers treatment of diffusive terms was either neglected or incorrect. This problem appeared to be a fundamental impediment, but was simply resolved in Hughes and Brooks[15]. In the context of the scalar advection–diffusion equation, our current perception of SUPG is consistent with Reference 15, although SUPG has been extended and refined for more general applications.

From the start, and continuing to this day, Claes Johnson and his colleagues have performed penetrating mathematical analyses of SUPG-type methods for various problems (see, e.g., References 16–23). In addition, important extensions of the methodology for the incompressible problem and unsteady case were made by the Göteborg team (see, e.g., References 18 and 19 respectively). Although my team's original contributions to this subject were for the most part intuitively based, I have more recently become very much influenced by the mathematical approach of Johnson. It seems to me now that significant progress in new areas requires at least a rudimentary appreciation of the mathematical underpinning of the methodology.

An outline of the subsequent sections of this paper follows. In Section 2 a summary of what is known about SUPG for scalar advection–diffusion problems is presented. Recent extensions, such as the 'discontinuity-capturing operator', are described. Brief mention is made in Section 3 of new developments for the incompressible problem. In Section 4 the current state of affairs for symmetric, coupled, multidimensional advective–diffusive systems is described. Attention is focused on the compressible Euler and Navier–Stokes equations in Section 5. It is shown that the concept of entropy variables plays a fundamental role in extending SUPG to systems of this type. Conclusions are drawn in Section 6.

2. SCALAR ADVECTION–DIFFUSION EQUATION

Consider the scalar advection–diffusion equation

$$\phi_{,t} + \mathbf{a}\cdot\nabla\phi = \nabla\cdot(\kappa\nabla\phi) + f, \tag{1}$$

where $\phi = \phi(\mathbf{x}, t)$ is the dependent variable (e.g., the temperature), $\mathbf{x} \in \Omega \subset \mathbf{R}^d$ is the spatial domain of the problem, d is the number of space dimensions, t is time, $\mathbf{a}$ is the velocity, κ is the diffusivity and f is the volumetric source term. An inferior comma denotes partial differentiation and ∇ denotes the spatial gradient operator. In what follows, to simplify the discussion, we will assume the flow is divergence-free, i.e.,

$$\nabla \cdot \mathbf{a} = 0 \tag{2}$$

and κ is constant. These assumptions may be dropped without essential alteration to the conclusions drawn.

If $\kappa > 0$, we have the *parabolic* case; if $\kappa = 0$, we have the *hyperbolic* case. The greatest challenge numerically is created by the case in which κ is positive but 'small' in the non-dimensional sense that the *element Peclet number* is large, viz.,

$$\alpha = \max \frac{|\mathbf{a}| h}{2\kappa} > 1, \tag{3}$$

where h is the finite element mesh parameter and $|\mathbf{a}|$ denotes the length of $\mathbf{a}$.

Well posed boundary conditions for the parabolic case consist of, for example, Dirichlet and diffusive-flux Neumann data:

$$\phi = g \quad \text{on} \quad \Gamma_g, \tag{4}$$

$$\mathbf{n} \cdot \kappa \nabla \phi = h \quad \text{on} \quad \Gamma_h, \tag{5}$$

where the boundary of Ω, denoted by Γ, admits the decomposition

$$\Gamma = \overline{\Gamma_g \cup \Gamma_h}, \tag{6}$$

$$\emptyset = \Gamma_g \cap \Gamma_h, \tag{7}$$

and $\mathbf{n}$ is the unit outward normal vector to Γ, and g and h are the given data. In the unsteady case an initial condition $\phi(\mathbf{x}, 0), \mathbf{x} \in \Omega$, need also be specified. For a while we will restrict our attention to the steady case, but later on return to recent developments applicable to the unsteady problem.

2.1. *Classical Galerkin finite element method*

In the finite element field the Galerkin variational formulation has dominated over the years. To define it we need to introduce a set of trial solutions, S^h, and a set of weighting functions, V^h. A distinguishing feature of the Galerkin method is that S^h and V^h are composed of the *same* class of functions, up to inhomogeneous Dirichlet boundary data. The Galerkin formulation of the advection–diffusion equation is given as follows. Find $\phi^h \in S^h$ such that for all $w^h \in V^h$,

$$B(w^h, \phi^h) = L(w^h), \tag{8}$$

where

$$B(w^h, \phi^h) = \int_\Omega (w^h \mathbf{a} \cdot \nabla \phi^h + \nabla w^h \cdot \kappa \nabla \phi^h) \, \mathrm{d}\Omega, \tag{9}$$

$$L(w^h) = \int_\Omega w^h f \, \mathrm{d}\Omega + \int_{\Gamma_h} w^h h \, \mathrm{d}\Gamma. \tag{10}$$

It is assumed that all trial solutions satisfy any Dirichlet conditions present. The diffusive-flux Neumann condition arises 'naturally' as a consequence of satisfaction of (8).

2.2. Error analysis

For any method of the form (8) we would like *stability* (also referred to as *coercivity*),

$$B(w^h, w^h) \geqslant c|||w^h|||^2, \tag{11}$$

where c is a constant and $|||\cdot|||$ is some norm, and *consistency*,

$$B(w^h, e) = 0, \tag{12}$$

where $e = \phi^h - \phi$ is the *error*. Stability, consistency and mild continuity requirements on the bilinear form $B(\cdot,\cdot)$ are all that is necessary to prove convergence in the $|||\cdot|||$ norm, viz.,

$$|||e||| \leqslant c(\phi)h^r, \tag{13}$$

where $r = r(k)$ increases with k, the order of complete polynomial contained in each element. Consistency is automatic for *weighted residual methods* in which the exact solution ϕ also satisfies the variational equation

$$B(w^h, \phi) = L(w^h). \tag{14}$$

Equation (12) follows from (8), (14) and the linearity of $B(\cdot,\cdot)$ with respect to the second argument. Galerkin's method is a weighted residual method and thus consistency is assured. However, stability is another matter. To see this in a simple setting, assume $\Gamma_h = \varnothing$ and we have homogenous Dirichlet boundary conditions. A direct calculation reveals

$$B(w^h, w^h) = \kappa \| \nabla w^h \|^2, \tag{15}$$

where $\|\cdot\|$ denotes the $L_2(\Omega)$ norm. This leads to an error estimate of the form

$$\|\nabla e\| = O((1 + \alpha)h^k). \tag{16}$$

The appearance of the element Peclet number α in the error estimate is a direct consequence of κ being the stability constant in (15). In the diffusion-dominated limit, α is small and (16) represents the usual optimal-order error estimate. On the other hand, in the advection-dominated case in which α is large, the appearance of α in (16) is a sign of trouble. In practice, spurious oscillations emanate from sharp layers and globally pollute numerical results (schematically illustrated in Figure 1). Similar phenomena occur in central finite difference approximations and are attributed to the weak coupling of odd- and even-numbered nodal equations. From a functional analysis standpoint, the problem derives from the poor stability provided by the 'small' constant κ in (15).

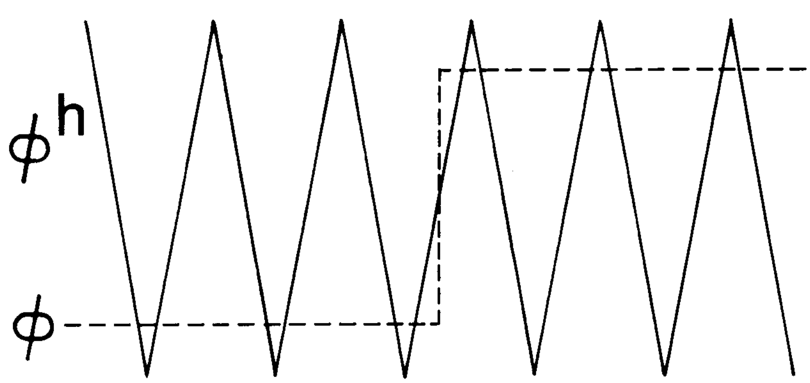

Figure 1.

2.3. *Classical artificial diffusion and upwind methods*

In order to improve upon the poor stability properties of central difference methods, artificial diffusion and upwind difference methods were created. Analogous finite element procedures can be developed by adding an artificial diffusion term to the Galerkin formulation:

$$\tilde{B}(w^h, \phi^h) = L(w^h), \tag{17}$$

where

$$\tilde{B}(w^h, \phi^h) = B(w^h, \phi^h) + \int_\Omega \nabla w^h \cdot \tilde{\kappa} \nabla \phi^h \, d\Omega \tag{18}$$

and $\tilde{\kappa}$ is an $O(h)$ artificial diffusivity coefficient. Stability is improved by the added term. Under the assumptions which led to (15) we now have

$$\tilde{B}(w^h, w^h) = \|(\kappa + \tilde{\kappa})^{1/2} \nabla w^h\|^2. \tag{19}$$

Now when κ is 'small', the $\tilde{\kappa}$ term, which is much larger, comes to the rescue, providing the necessary dose of stability. Unfortunately, the consistency property, (12), is now violated because (17) is no longer a weighted residual method, i.e., the exact solution does *not* satisfy (17). The upshot is that the method is no better than first-order accurate independent of the order of complete polynomial present in the finite element interpolations. Thus stability is achieved at the expense of accuracy and typical results exhibit overly diffuse behaviour (see Figure 2 for a schematic illustration). It may be concluded that neither the Galerkin method nor the classical artificial diffusion/upwind method possesses the requisite mathematical properties necessary for achieving good behaviour in practice. Much research effort has gone into the development of alternative procedures with the aim of simultaneously attaining good stability and accuracy properties.

2.4. *Streamline–upwind/Petrov–Galerkin method ('SUPG')*

SUPG may also be viewed as a modification to the classical Galerkin method. The physical idea is to increase control over the advective-derivative term. This can be done by adding an artificial diffusion term which acts only in the streamline direction. The key idea in SUPG, which distinguishes it from classical streamline–upwind difference methods, for example, is that this stabilizing control can be introduced within a weighted residual format, thus maintaining consistency. The method is defined by

$$B_\tau(w^h, \phi^h) = L_\tau(w^h), \tag{20}$$

where

$$B_\tau(w^h, \phi^h) = B(w^h, \phi^h) + \sum_{e=1}^{n_{el}} \int_{\Omega^e} \tau \mathbf{a} \cdot \nabla w^h (\mathbf{a} \cdot \nabla \phi^h - \nabla \cdot \kappa \nabla \phi^h) \, d\Omega, \tag{21}$$

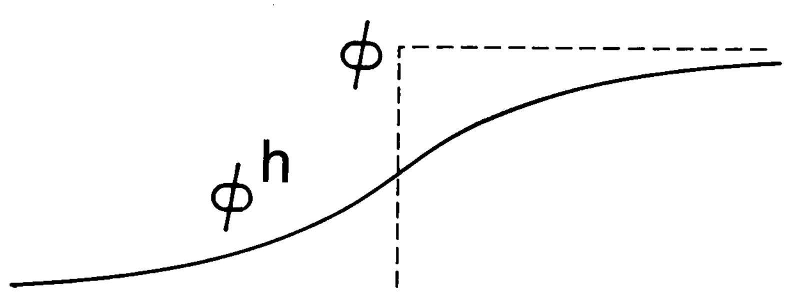

Figure 2.

$$L_\tau(w^h) = L(w^h) + \sum_{e=1}^{n_{el}} \int_{\Omega^e} \tau \mathbf{a} \cdot \nabla w^h f \, d\Omega, \tag{22}$$

$\Omega^e \subset \Omega$ is the domain of the eth element, n_{el} is the number of elements and τ is a locally defined parameter having dimensions of time which we refer to as the *intrinsic time scale*. Note that the additional terms are integrals over *element interiors*. Furthermore, consistency holds for all τ, i.e.,

$$B_\tau(w^h, e) = 0. \tag{23}$$

At the same time stability is enhanced in that if locally $\tau = O(h/|\mathbf{a}|)$ in the advection-dominated case, and $\tau = O(h^2/\kappa)$ in the diffusion-dominated case, then*:

advection-dominated case

$$\|e\| \leqslant c(\phi) h^{k+(1/2)}, \tag{25}$$

$$\|\mathbf{a} \cdot \nabla e\| \leqslant c(\phi) h^k; \tag{26}$$

diffusion-dominated case

$$\|e\| \leqslant c(\phi) h^{k+1}, \tag{27}$$

$$\|\nabla e\| \leqslant c(\phi) h^k. \tag{28}$$

All are optimal except (25) which is near optimal ('gap 1/2'). Furthermore, Johnson and his colleagues have derived localization results for rough solutions possessing sharp internal and boundary layers. These results show that the above error estimates hold on Ω modulo a small neighborhood of the layers. Consequently, SUPG is seen to be a robust methodology and one possessing higher-order accuracy (the order of accuracy depends only on k, as may be seen from (25)–(28)).

By virtue of the fact that SUPG is a higher-order accurate linear method, monotone approximations of sharp layers are not possible. Thus some overshoot and/or undershoot will appear in these circumstances. The localization results guarantee that these will not globally pollute the solution as is the case of the classical Galerkin method for which no localization results are possible. These observations are confirmed by numerical experience (see Figure 3 for a schematic representation of response). The only way to simultaneously achieve high-order accuracy and smooth approximations to sharp layers is to introduce non-linear mechanisms. A particular variant of this theme will be described subsequently.

The appellation 'streamline–upwind/...' derives from the following facts. The difference stencils produced by the method are *centred* about points shifted upwind along, and aligned with, streamlines. When written in *Euler–Lagrange form* the weighting function on element interiors involves the τ-term, viz.,

$$0 = \sum_{e=1}^{n_{el}} \int_{\Omega^e} \tilde{w}^h (\mathbf{a} \cdot \nabla \phi^h - \nabla \cdot \kappa \nabla \phi^h - f) \, d\Omega$$

$$+ \text{diffusive-flux inter-element continuity and boundary terms,} \tag{29}$$

where

$$\tilde{w}^h = w^h + \tau \mathbf{a} \cdot \nabla \phi^h. \tag{30}$$

* Functional analysis techniques and exact solutions of difference equations for simple model problems yield results which are in agreement regarding the form of τ. The 'best' computational formula for τ is only known for very simple cases. However, effective formulae are available for general situations. See Reference 24 for our current thinking on this matter. A deeper understanding of the process for selecting τ would be welcome.

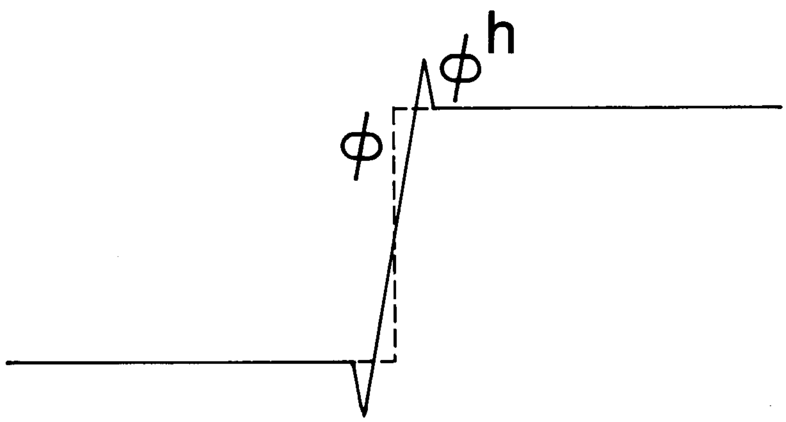

Figure 3.

The diffusive-flux inter-element continuity and boundary terms are the same as for the classical Galerkin method. Petrov's name comes in whenever $\tilde{w}^h \neq w^h$, although the reason for this attribution still seems to us obscure.

SUPG first appeared in the form (20), (21) in Reference 15. The continuity requirements of trial and weighting functions and whether or not SUPG is a conforming or non-conforming method have caused considerable confusion among individuals who do not understand the mathematical basis of it: *SUPG requires only C^0-continuous finite element interpolations and it is correctly classified as a conforming method.*

Raithby's difference approach (References 25, 26) would amount to ignoring the diffusion and source terms in the element integrals in (21) and (22) respectively. This violates the weighted residual recipe and leads to pathological behaviour numerically (see References 15 and 27).

2.5. Discontinuous Galerkin

The discontinuous Galerkin method is the only other finite element method which has been mathematically analysed to the extent SUPG has. It seems to have been originally conceived by Reed and Hill[28] in the context of neutron transport problems. The mathematical analysis of the method commenced in Lesaint and Raviart[29] and was continued and improved upon by Johnson and his colleagues (see, e.g., Reference 18). In its primitive form, the discontinuous Galerkin method is restricted to the case of pure advection (i.e., $\kappa = 0$). Trial solutions and weighting functions are taken to be *discontinuous* across inter-element boundaries. Continuity of flux is enforced weakly by terms appended to the classical Galerkin bilinear form. The method is given by

$$B_\Gamma(w^h, \phi^h) = L(w^h), \tag{31}$$

where

$$B_\Gamma(w^h, \phi^h) = \int_\Omega w^h \mathbf{a} \cdot \nabla \phi^h \, d\Omega - \sum_{e=1}^{n_{el}} \int_{\Gamma^e_{\text{inflow}}} w^h_+ \mathbf{n} \cdot \mathbf{a} (\phi^h_+ - \phi^h_-) \, d\Gamma. \tag{32}$$

The inflow boundary of element e, denoted by Γ^e_{inflow}, is defined to be that part of Γ^e where $\mathbf{n} \cdot \mathbf{a} < 0$. This idea and the meaning of the + and − subscripts are schematically depicted in Figure 4.

The discontinuous Galerkin method shares with SUPG the same error estimates and localization results. The generalization to hyperbolic systems requires the definition of a numerical flux vector on inter-element boundaries which is a function of the + and − state vectors. Several

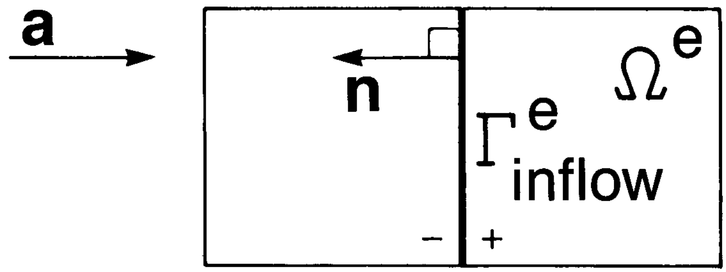

Figure 4.

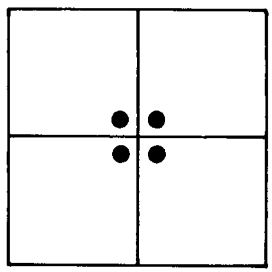

Figure 5.

suitable candidates emanate from the finite difference literature such as the well known Godunov and Osher numerical fluxes. The case of piecewise constants (i.e., $k = 0$) seems essentially identical to what is referred to as the 'finite volume method'. Chavent, Jaffre and their colleagues have been actively pursuing the development of discontinuous Galerkin methods for petroleum reservoir simulation (see, e.g., References 30 and 31). The generalization to cases in which diffusion is present requires mixed finite element approximations and has not been actively pursued as far as we are aware. A main practical impediment to the adoption of discontinuous Galerkin methods is that for $k \geqslant 1$ the dimensions of trial and weighting function spaces become enormous. This is illustrated schematically for bilinear quadrilaterals in Figure 5. The situation is worse for triangles and becomes even more unfavourable in three dimensions for hexahedra and tetrahedra (e.g., the ratio of degrees of freedom for linear discontinuous tetrahedra to linear continuous tetrahedra is 20!).

2.6. Space–time finite element formulations

Johnson and colleagues have proposed using discontinuous Galerkin in time with either SUPG or discontinuous Galerkin in space (see, e.g., Reference 18). This leads to fully discrete implicit systems in which the solution in the space–time slab $\Omega \times]t_n, t_{n=1}[$ depends only upon the previously obtained solution at time t_n^-. Error estimates of the form (25)–(28) hold as well for this case, where h is now a space–time mesh parameter. This is a major contribution which should forever dispel the prevalent myth in finite difference circle that somehow finite elements are not appropriate for hyperbolic problems.

2.7. Discontinuity capturing

In order to improve upon the ability of SUPG to smoothly resolve sharp layers, a discontinuity-

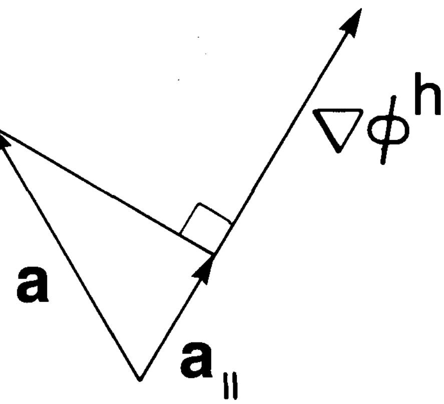

Figure 6.

capturing operator was introduced by Hughes *et al.*[24] In this case the modified weighting function is written as (cf. Reference 32)

$$\tilde{w}^h = w^h + \tau_1 \mathbf{a}\cdot\nabla w^h + \tau_2 \mathbf{a}_\parallel\cdot\nabla w^h \tag{33}$$

where $\mathbf{a}_\parallel$ is the projection of $\mathbf{a}$ on $\nabla\phi^h$, i.e.,

$$\mathbf{a}_\parallel = \frac{\mathbf{a}\cdot\nabla\phi^h}{|\nabla\phi^h|^2}\nabla\phi^h \tag{34}$$

(See Figure 6.) This method is seen to be *non-linear* in that $\mathbf{a}_\parallel = \mathbf{a}_\parallel(\nabla\phi^h)$. The interaction of this weighting function with the advection term in the residual may be decomposed into three contributions:

$$\begin{aligned}\tilde{w}^h(\cdots + \mathbf{a}\cdot\nabla\phi^h + \cdots) = \cdots + \underbrace{w^h\mathbf{a}\cdot\nabla\phi^h}_{\text{Galerkin term}} + \underbrace{\nabla w^h\cdot\tau_1\mathbf{a}\mathbf{a}^{\mathrm{T}}\nabla\phi^h}_{\text{streamline operator}} \\ + \underbrace{\nabla w^h\cdot\tau_2\mathbf{a}_\parallel\mathbf{a}_\parallel^{\mathrm{T}}\nabla\phi^h}_{\text{discontinuity-capturing operator}} + \cdots.\end{aligned} \tag{35}$$

In obtaining (35) we have used the result

$$\mathbf{a}\cdot\nabla\phi^h = \mathbf{a}_\parallel\cdot\nabla\phi^h \tag{36}$$

which follows from (34). The streamline matrix $\mathbf{a}\mathbf{a}^{\mathrm{T}}$ is a rank-1 positive-semidefinite matrix which acts only in the streamline direction. The discontinuity-capturing matrix $\mathbf{a}_\parallel\mathbf{a}_\parallel^{\mathrm{T}}$ is likewise a rank-1 positive-semidefinite matrix which acts only in the direction of the discrete solution gradient and thus provides an ingredient with the potential for controlling spurious oscillations in the discrete solution. Stability is clearly enhanced by the presence of the discontinuity-capturing term, and concomitantly consistency is maintained (see (23)). Nevertheless, due to the non-linear nature of discontinuity capturing its error analysis is an open problem. See Tezduyar and Park[33] for refinements and Johnson and Szepessy[22] for space–time generalizations. The generalization of the discontinuity-capturing operator to systems has proven to be an essential ingredient for accurately capturing shock waves in our formulation of the compressible Euler and Navier–Stokes equations (see Section 2.4).

Remark. The inspiration behind the development of the discontinuity-capturing operator emanates from finite difference ideas of Davis.[34]

3. INCOMPRESSIBLE FLOWS

SUPG was originally developed as a means of improving upon the classical Galerkin formulation of the incompressible Navier–Stokes equations. The advection–diffusion equation was primarily used as a convenient model equation because of its linearity and greater simplicity. The original SUPG formulations intuitively adopted analogous weighting functions to those developed for the advection–diffusion model problems. Practical calculations exhibited good accuracy and a greater degree of robustness than the classical Galerkin formulation (see, e.g., Brooks and Hughes[27]). From the mathematical point of view one needs to account for the pressure gradient in the weighting function as shown by Johnson and Saranen,[19] who analyse the fully discrete case for divergence-free basis functions. Hansbo[35] has obtained good numerical results with this formulation. Progress has recently been made without the divergence-free hypothesis (Johnson, private communication, 1986). A convergent formulation of the Stokes problem has recently been developed in which a pressure gradient term appears in the weighting function for the momentum residual (see Hughes *et al.*[36]). In this formulation it is not necessary to satisfy the Babuška–Brezzi condition and, in fact, all combinations of continuous velocity and pressure interpolations are convergent. See Brezzi and Douglas[37] for further analysis of the method proposed in Reference 36 and generalizations.

SUPG-type methods for incompressible problems have now gone beyond the original aspiration of simply stabilizing convective terms, as evidenced by the reference cited.

4. ADVECTIVE–DIFFUSIVE SYSTEMS

Physical advective–diffusive systems are usually written in the form

$$\mathbf{U}_{,t} + \mathbf{A}\cdot\nabla\mathbf{U} = \nabla\cdot(\mathbf{K}\,\nabla\mathbf{U}) + f, \tag{37}$$

where

$$\mathbf{U} = \begin{Bmatrix} U_1 \\ U_2 \\ \vdots \\ U_m \end{Bmatrix}, \quad \nabla\mathbf{U} = \begin{Bmatrix} \mathbf{U}_{,1} \\ \vdots \\ \mathbf{U}_{,d} \end{Bmatrix}, \quad \mathbf{A} = \begin{bmatrix} \mathbf{A}_1 \\ \vdots \\ \mathbf{A}_d \end{bmatrix}, \tag{38}$$

$$\mathbf{A}\cdot\nabla\mathbf{U} = \mathbf{A}_1\mathbf{U}_{,1} + \ldots + \mathbf{A}_d\mathbf{U}_{,d}, \tag{39}$$

$$\mathbf{K} = \begin{bmatrix} \mathbf{K}_{11} & \ldots & \mathbf{K}_{1d} \\ \vdots & & \vdots \\ \mathbf{K}_{d1} & \ldots & \mathbf{K}_{dd} \end{bmatrix}, \tag{40}$$

$$\nabla\cdot(\mathbf{K}\,\nabla\mathbf{U}) = \sum_{i,j=1}^{d} (\mathbf{K}_{ij}\mathbf{U}_{,j})_{,i}, \tag{41}$$

where each $\mathbf{A}_i$ and $\mathbf{K}_{ij}$ is an $m \times m$ matrix. We refer to $\mathbf{A}\cdot\nabla\mathbf{U}$ as the *generalized advection term* and $\nabla\cdot(\mathbf{K}\,\nabla\mathbf{U})$ as the *generalized diffusion term*. For example, the compressible Navier–Stokes equations can be written in the form (37), where $\mathbf{U}$ represents the *conservation variables*.

For many physical systems a change of variables $\mathbf{U} = \mathbf{U}(\mathbf{V})$ exists such that (37) can be written as

$$\mathbf{A}_0\mathbf{V}_{,t} + \tilde{\mathbf{A}}\cdot\nabla\mathbf{V} = \nabla\cdot(\tilde{\mathbf{K}}\nabla\mathbf{V}) + F, \tag{42}$$

where the $\tilde{\mathbf{A}}_i$ are symmetric ($1 \leqslant i \leqslant d$), $\mathbf{A}_0$ is symmetric and positive-definite and $\tilde{\mathbf{K}}$ is symmetric and positive-semidefinite. If $\tilde{\mathbf{K}} = \mathbf{0}$, then (42) is called a *symmetric hyperbolic system* or a *Friedrichs' system* (e.g., the compressible Euler equations). If $\tilde{\mathbf{K}}$ is positive-definite, then (42) is called a *symmetric parabolic system.* If $\tilde{\mathbf{K}}$ is positive-semidefinite, but not positive-definite, then (42) is called a *symmetric incompletely parabolic system* (e.g., the compressible Navier–Stokes equations).

For *linear* systems of the form (42), a generalization of SUPG is available. For the hyperbolic case, Johnson *et al.*[18] derived error estimates for the space–time formulation analogous to (25), (26) and localization results. Similar results also hold for discontinuous Galerkin.[18] The parabolic case, accounting for arbitrary dominance of advection or diffusion in the various 'modes' of the system, turns out to be delicate. The generalization of SUPG for the steady case is considered by Hughes and Mallet[38] and the space–time formulation for the unsteady case is analysed in Hughes *et al.*[39] At the time of this writing, what constitute well posed boundary conditions for the general incompletely parabolic case are not known. Once these are established, it is anticipated that the methods of References 38 and 39 will suffice to establish convergence and error estimates.

5. COMPRESSIBLE EULER AND NAVIER–STOKES EQUATIONS

Assuming sufficient smoothness, the compressible Navier–Stokes equations can be written in quasi-linear form (37) in terms of conservation variables

$$\mathbf{U} = \begin{Bmatrix} \rho \\ \rho\mathbf{u} \\ \rho e \end{Bmatrix}, \tag{43}$$

where ρ is the density, $\mathbf{u}$ is the velocity vector and e is the total energy. Classical $L_2(\Omega)$ stability estimates for systems of the type (37) are derived by simply taking the dot product of (37) with $\mathbf{U}$ (see, e.g., Courant and Hilbert[40]). In the present case this does not even make dimensional sense, as may be seen from the first term:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}[\rho^2\underbrace{(1+|\mathbf{u}|^2+e^2)}_{???}] + \cdots. \tag{44}$$

This suggest that the $L_2(\Omega)$ inner product structure is inappropriate for the compressible Navier–Stokes equations and consequently so would be a classical-type Galerkin formulation.

Remark. For attempts at developing SUPG-type methods within this framework see References 32, 41 and 42.

5.1. *Entropy variables*

Entropy variables have been investigated by Godunov,[43] Mock,[44] Harten,[45] Tadmor,[46] Dutt[47] and Hughes *et al.*[48] Let

$$H = H(\mathbf{U}) = -\rho s, \tag{45}$$

where

$$s = \ln\left[\frac{p}{p_o}\left(\frac{\rho_o}{\rho}\right)^{\gamma}\right] \tag{46}$$

is the non-dimensional entropy, p is the pressure, a subscript zero represents a reference value and γ is the ratio of specific heats (assumed constant). Entropy variables are defined by

$$\mathbf{V} = \partial H/\partial \mathbf{U}. \tag{47}$$

By virtue of the fact that H is a convex function of $\mathbf{U}$, (47) leads to a well defined change of variables, $\mathbf{U} = \mathbf{U}(\mathbf{V})$. Employing this change of variables in (37) results in form (42) along with the stated symmetry and definiteness properties of the arrays. Furthermore, taking the dot product of (42) with $\mathbf{V}$ not only makes dimensional sense, but additionally makes physical sense in that the *Clausius–Duhem inequality*, or *second law of thermodynamics*, ensues:

$$\begin{aligned} 0 &= \mathbf{V}\cdot(\mathbf{A}_0\mathbf{V}_{,t} + \tilde{\mathbf{A}}\cdot\nabla\mathbf{V} - \nabla\cdot(\tilde{\mathbf{K}}\nabla\mathbf{V}) - F) \\ &= \frac{1}{c_v}\left[-(\rho\eta)_{,t} - \nabla\cdot(\rho\eta\mathbf{u}) + c_v\nabla\mathbf{V}\cdot(\tilde{\mathbf{K}}\nabla\mathbf{V}) - \nabla\cdot\left(\frac{\mathbf{q}}{\theta}\right) + \frac{\rho r}{\theta}\right], \end{aligned} \tag{48}$$

implying

$$(\rho\eta)_{,t} + \nabla\cdot(\rho\eta\mathbf{u}) + \nabla\cdot\left(\frac{\mathbf{q}}{\theta}\right) - \frac{\rho r}{\theta} \geqslant 0, \tag{49}$$

where c_v is the specific heat at constant volume, $\eta = c_v s$ is the thermodynamic entropy, θ is the temperature, $\mathbf{q}$ is the heat flux vector and r is the radiation per unit mass. This is the basic non-linear stability condition for the compressible Navier–Stokes equations. An important result follows: *a classical Galerkin formulation inherits the entropy production property.*[48] This can be seen from

$$\begin{aligned} 0 = \sum_{e=1}^{n_{el}} \int_{\Omega^e} \mathbf{W}^h\cdot[\mathbf{A}_0\mathbf{V}^h_{,t} + \tilde{\mathbf{A}}\cdot\nabla\mathbf{V}^h - \nabla\cdot(\tilde{\mathbf{K}}\nabla\mathbf{V}^h) - F]\,d\Omega \\ + \text{continuity and boundary terms.} \end{aligned} \tag{50}$$

Replacing $\mathbf{W}^h$ by $\mathbf{V}^h$ in (50) and proceeding as in (48), (49) leads to a global statement of (49) in terms of the discrete solution $\mathbf{V}^h$. This holds for the fully discrete space–time formulation as well as the semidiscrete t-continuous formulation. This is a good start, but, as is apparent from linear analysis, not enough. Another way of appreciating this fact is to consider the compressible Euler equations and assume C^0-continuous basis functions are employed. Then proceeding as indicated above yields

$$\begin{aligned} 0 &= \int_{\Omega} \mathbf{V}^h\cdot(\mathbf{A}_0\mathbf{V}^h_{,t} + \tilde{\mathbf{A}}\cdot\nabla\mathbf{V}^h)\,d\Omega \\ &= \frac{1}{c_v}\int_{\Omega} [(\rho\eta)_{,t} + \nabla\cdot(\rho\eta\mathbf{u})]\,d\Omega. \end{aligned} \tag{51}$$

In words, entropy is always conserved*, even in the presence of shocks in the exact solution, making convergence in this case impossible. To correct this and other deficiencies of the classical Galerkin method, a generalized SUPG-type formulation with discontinuity capturing may be adopted.

Remark. Mazet and colleagues have developed a finite element method for hyperbolic conservation laws based upon extremizing the rate of entropy production (see References 49–52 for details).

5.2. SUPG

We give here only a brief sketch of the essential features of our SUPG-type formulation for

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* For subsonic flows, entropy conservation is of course appropriate. This is an interesting property of the Galerkin method which should be exploitable in practice.

the compressible Euler and Navier–Stokes equations. The original sources may be consulted for more details (see References 38, 39, 48, 53–56). The variational equation takes the form

$$0=\sum_{e=1}^{n_{el}}\int_{\Omega^e}\tilde{\mathbf{W}}^h\cdot(\mathbf{A}_0\mathbf{V}^h_{,t}+\tilde{\mathbf{A}}\cdot\nabla\mathbf{V}^h-\nabla\cdot(\tilde{\mathbf{K}}\nabla\mathbf{V}^h)-F)\,\mathrm{d}\Omega+\text{continuity and boundary terms,} \quad (52)$$

where

$$\tilde{\mathbf{W}}^h=\mathbf{W}^h+\mathbf{T}_1\cdot\nabla\mathbf{W}^h+\mathbf{T}_2\cdot\nabla\mathbf{W}^h, \quad (53)$$

in which $\mathbf{T}_1$ is the generalized 'streamline' matrix for systems (in fact, it has nothing to do with streamlines; see Reference 38) and $\mathbf{T}_2$ is the generalized discontinuity-capturing matrix for systems. The latter matrix always has rank 1 and is proportional to the projection of $\mathbf{T}_1$ on the direction $\nabla\mathbf{V}^h$ with respect to the Riemannian metric $\mathbf{A}_0$ (see Reference 55). Note that (53) applies to any system, linear or non-linear, that can be expressed in the form (42). The fully discrete variational equation is identical except we need to work on space–time slabs and add the weakly enforced t-continuity term (see References 18 and 39). Johnson and Szepessy have made progress analysing SUPG in the non-linear hyperbolic case (see References 20–22). They show in Reference 22 that if the finite element solutions converge as $h\to 0$, then they converge to an entropy solution. For Burgers' equation in one dimension they establish the stronger result that if the finite element solutions remain uniformly bounded as $h\to 0$, then convergence to an entropy solution is attained. These results hold with or without the discontinuity-capturing term. However, superior numerical results are obtained when it is employed.[22] The mathematical results of Johnson and Szepessy have thus made precise the assertion that entropy variables preserve weak solutions. We have obtained good numerical results with our approach on a variety of compressible Euler and Navier–Stokes problems (see, e.g., References 53, 54, 56–58).

6. CONCLUSIONS

The numerical solution of many practical problems of fluid flow is still far from a reality. Improved schemes are needed and this topic is likely to engage the attention of researchers for quite some time. SUPG-type methods have been successfully applied to a variety of flow problems including scalar advection–diffusion processes and incompressible viscous flows. A theory has evolved which is applicable to general linear, coupled, multidimensional advective–diffusive systems and non-linear systems possessing an entropy function. Steady as well as unsteady space–time SUPG methods exist of all orders of accuracy. At the same time, good stability properties are incorporated in SUPG. An impressive number of mathematical results have already been derived and progress continues to be made.

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